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A Korovkin Theorem for Abstract Lebesgue Spaces

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Wulbert and Meir have each obtained a Korovkin theorem for weak convergence of operators on an L_1 space. Here we prove a result which includes both of these theorems and which provides a general setting for weak Korovkin type L_1 convergence of operators which are not assumed positive. © 2000 Academic Press
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1. INTRODUCTION

Wulbert [4] and Meir [2] have obtained Korovkin type results on the weak convergence of a sequence of contraction linear operators on $L_1[0, 1]$. In this paper we derive a theorem which includes both of these results. It is embedded in a more general setting, but in the case $L_1[0, 1]$ considered by Wulbert and Meir, we will show

THEOREM 1.1. *Let (T_n) be a sequence of contraction linear operators on $L_1[0, 1]$. If*

(i) $T_n 1 \xrightarrow{s} 1$ *and*
(ii) $T_n f \xrightarrow{w} f$ *for the function* $f(x) = x$, *then*

$$T_n f \xrightarrow{s} f \qquad \text{for all} \quad f \text{ in } L_1[0, 1].$$

(Here, s and w denote strong and weak convergence respectively.)

Meir obtained a similar result assuming that the operators are positive while Wulbert assumed the weaker condition (ii), that $T_n f \xrightarrow{w} f$ for the two functions $f(x) = x$ *and* $f(x) = x^2$.

Furthermore to allow our results to apply to more general L_1 spaces including l^1 spaces as well as $L_1[0, 1]$, we replace $L_1[0, 1]$ by an Abstract Lebesgue (AL) space, and the constant function 1 by a generalized weak unit. More generally then we will prove the following

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THEOREM 1.2. *Let E be an AL space with generalized weak unit (e_α) and let (T_n) be a sequence of contraction linear operators on E such that $T_n e_\alpha \xrightarrow{s} e_\alpha$ for all α. Let $N = \{f \colon T_n f \xrightarrow{w} f\}$.*
Then N is a closed sublattice of E and

(i) $T_n f \xrightarrow{s} f$ *and*
(ii) $\tau_n f \xrightarrow{s} f$

for all $f \in N$.

(Here τ_n is the modulus $|T_n|$ as defined in the next section.)

Theorem 1.1 follows at once by noting that the smallest closed sublattice of $L_1[0, 1]$ containing 1 and the function $f(x) = x$ is all of $L_1[0, 1]$.

2. NOTATION

We refer the reader to Birkhoff ([1], Ch. XV) or Schaefer ([3], Ch. 2) for the basic ideas of Banach lattices.

DEFINITION 2.1. A (real) Banach lattice is called an AL space if

$$\|x + y\| = \|x\| + \|y\| \qquad \text{for all} \quad x, y \geqslant 0.$$

For E an AL space define $E^+ = \{x \colon x \geqslant 0\}$ and E_1 the unit ball in E with similar definitions for E^*—the dual space of E.

An orthogonal (or disjoint) system ([3], p. 50) in E is a subset S of $E \backslash \{0\}$ such that $u \wedge v = 0$ for all distinct u, v in S. A generalized weak unit $\{e_\alpha\}$ for E is a maximal orthogonal family in E^+. Such families clearly exist via Zorn's lemma. Without loss of generality we will also assume that the family is normalised so that $\|e_\alpha\| \equiv 1$.

The following properties of AL spaces will be needed.

A subset $A \subset E$ is weakly sequentially precompact (wsp) if every sequence in A has a weakly Cauchy subsequence. Since E is weakly sequentially complete ([3], p. 119), "Cauchy" can be replaced by "convergent." If A is norm bounded, then ([3], p. 152) A is wsp iff for all disjoint majorized sequences (ψ_n) in E^{*+},

$$\lim_n \sup_{x \in A} \langle |x|, \psi_n \rangle = 0. \tag{2.1}$$

The map from $E^+ \to \mathbf{R}$ given by $x \to \|x\|$ extends to define a linear functional $\psi_0 \in E^*$—the evaluating functional.

Finally if T is a linear operator on E, the modulus $|T|$ can be defined by

$$|T|\,x = \sup_{|y| \leqslant x} |Ty|, \qquad x \in E^+$$

which extends to a linear operator on E satisfying

$$-|T| \leqslant T \leqslant |T| \qquad \text{and} \qquad \|\,|T|\,\| = \|T\|$$

(see [3], Chapter IV, Section 1, especially Corollary 2).

For notational simplicity we write τ for $|T|$.

3. THE KOROVKIN THEOREM

Throughout, E denotes an AL space with generalized unit $\{e_\alpha\}$. (T_n) is a sequence of contraction linear operators on E with the moduli of T_n denoted by τ_n. Let $N = \{f\colon T_n f \xrightarrow{w} f\}$. Clearly N is a (closed) subspace of E. To prove our Korovkin theorem stated above we first need the following results

Lemma 3.1. *Suppose* $u \geqslant 0$ *and that* $T_n u \xrightarrow{w} u$. *Then*

(i) $|T_n u| \xrightarrow{w} u$ *and*

(ii) $\tau_n u \xrightarrow{w} u$.

Proof. (i) Since $T_n u \xrightarrow{w} u$, $\{T_n u\}$ is wsp and hence so too is $\{|T_n u|\}$. (This follows e.g., from Eq. (2.1).)

So we can choose a subsequence $(T_{n(j)} u)$ such that $|T_{n(j)} u| \xrightarrow{w} v$ (say). Clearly $v \geqslant u$.

Using the evaluating functional we have $\|T_{n(j)} u\| \to \|v\|$.

$$\text{I.e., } \|v\| = \lim_j \|T_{n(j)} u\| \leqslant \|u\|.$$

This combined with $v \geqslant u$ and the AL property shows that $v = u$. Applying this argument to any subsequence of $(T_n u)$, we have $|T_n u| \xrightarrow{w} u$.

Note that this implies, via the evaluating functional, that $\|\,|T_n u|\,\| \to \|u\|$.

(ii) For $\psi \in E_1^*$ we have

$$|\langle \tau_n u - u, \psi \rangle| \leqslant |\langle \tau_n u - |T_n u|, \psi \rangle| + |\langle |T_n u| - u, \psi \rangle|.$$

The second term $\to 0$ by (i) and the first term is bounded by

$$\begin{aligned} \|\tau_n u - |T_n u|\| &= \|\tau_n u\| - \| |T_n u| \| && \text{by the AL property} \\ &\leqslant \|u\| - \| |T_n u| \| && \text{as } \tau_n \text{ is a contraction} \\ &\to 0 && \text{by (i).} \end{aligned}$$

So $\tau_n u \xrightarrow{w} u$.

PROPOSITION 3.2. *Suppose that $T_n e_\alpha \xrightarrow{w} e_\alpha$ for all α. Then*

(i) *N is a sublattice of E and*

(ii) *$\tau_n f \xrightarrow{w} f$ for all $f \in N$.*

Proof. (i) It suffices to show that if $f \in N$ then so does $|f|$.

Fix $f \in N$. We firstly reduce the problem to the case where a weak unit for E exists.

Let spt $f = \{\alpha : |f| \wedge e_\alpha > 0\}$.

Since the e'_αs are disjoint and positive we have for any $\alpha_1, \alpha_2, ..., \alpha_n$,

$$\|f\| \geqslant \sum_{i=1}^{n} \||f| \wedge e_{\alpha_i}\|$$

so that spt f is countable.

Define

$$e = \sum \frac{e_n}{2^n \|e_n\|}$$

with summation over spt f.

For $A \subset E$ let $\perp A = \{x : |x| \perp |y| \text{ for all } y \in A\}$.

Then ([1], p. 309), $\perp\perp(e)$ is a sub AL space of E for which e is a weak unit. Furthermore since

$$|f| = \sup_{n, H} \sum_{\alpha \in H} (|f| \wedge ne_\alpha)$$

where H runs through all finite subsets of spt f (see e.g., [3], p. 55, Proposition 1.9), $\perp\perp(e)$ also contains f.

Hence we can assume that E has a weak unit $e > 0$.

Clearly $T_n e \xrightarrow{w} e$ and so by Lemma 3.1 $\tau_n e \xrightarrow{w} e$. We now show that $T_n |f| \xrightarrow{w} |f|$.

For $m=1, 2\ldots$ we have $0 \leqslant |f| \wedge me \leqslant me$ so that

$$\tau_n(|f| \wedge me) \leqslant m\tau_n e.$$

Since $\tau_n e$ is weakly convergent, $\tau_n(|f| \wedge me)$ is wsp for each fixed m.

Let $m=1$ and choose a subsequence $n(1, j)$ such that

$$\tau_{n(1, j)}(|f| \wedge e) \xrightarrow{w} g_1 \qquad \text{(say)}.$$

Now choose a subsequence $n(2, j)$ of $n(1, j)$ such that

$$\tau_{n(2, j)}(|f| \wedge 2e) \xrightarrow{w} g_2 \qquad \text{etc.}$$

Diagonalization now yields a sequence $n(j, j)$ such that

$$\tau_{n(j, j)}(|f| \wedge me) \xrightarrow{w} g_m \qquad \text{for each } m.$$

Clearly (g_m) is increasing and via the evaluating functional we see that

$$\|g_m\| = \lim_j \|\tau_{n(j, j)}(|f| \wedge me)\| \leqslant \|f\|.$$

So (g_m) converges (order and strongly) to g say and $\|g\| \leqslant \|f\|$. (See e.g., [3] Proposition 8.2.)

Further for $\psi \in E^*$ we have

$$\begin{aligned} |\langle \tau_{n(j, j)} |f| - g, \psi \rangle| \leqslant\ & |\langle \tau_{n(j, j)}(|f| - |f| \wedge me), \psi \rangle| \\ & + |\langle \tau_{n(j, j)}(|f| \wedge me) - g_m, \psi \rangle| \\ & + |\langle g_m - g, \psi \rangle|. \end{aligned}$$

The first term on the right is bounded by $\| |f| - |f| \wedge me\| \, \|\psi\|$ which is small for large m, as e is a weak unit. Similarly the third term is small for m sufficiently large. Finally for fixed m, the second term is small for large j. We deduce that

$$\tau_{n(j, j)} |f| \xrightarrow{w} g.$$

But also

$$\tau_{n(j, j)} |f| \geqslant |T_{n(j, j)} f| \geqslant T_{n(j, j)} f$$

and in the limit we then have $g \geqslant f$. Similarly $g \geqslant -f$ so that $g \geqslant |f|$. This together with $\|g\| \leqslant \|f\|$ and the AL property shows that $g = |f|$. Now applying this reasoning to an arbitrary subsequence of $(T_n f)$ we obtain $\tau_n |f| \xrightarrow{w} |f|$.

To show now that $T_n |f| \xrightarrow{w} |f|$, we first notice that since $|T_n(|f| \wedge me)| \leqslant \tau_n(|f| \wedge me)$ then for each fixed m, $\{T_n(|f| \wedge me)\}$ is wsp. By the argument above, there exists a subsequence $(T_{n(j,j)})$ and a sequence (h_m) such that

$$T_{n(j,j)}(|f| \wedge me) \xrightarrow{w} h_m \qquad \text{for all } m.$$

Fix $\psi \in E^{*+}$. Then

$$\begin{aligned} 0 &\leqslant \langle (\tau_{n(j,j)} - T_{n(j,j)}) |f|, \psi \rangle \\ &\leqslant \langle (\tau_{n(j,j)} - T_{n(j,j)})(|f| - |f| \wedge me), \psi \rangle \\ &\quad + \langle (\tau_{n(j,j)} - T_{n(j,j)}) me, \psi \rangle. \end{aligned}$$

The first term on the right can be made small by choosing m large and for fixed large m the second term converges to 0 as $j \to \infty$. We deduce that $(\tau_{n(j,j)} - T_{n(j,j)}) |f| \xrightarrow{w} 0$ and hence that

$$T_{n(j,j)} |f| \xrightarrow{w} |f|.$$

Applying this to any subsequence of $(T_n |f|)$ we have that

$$T_n |f| \xrightarrow{w} |f|.$$

(ii) If $f \in N$ then from Eq. (3.1) $\tau_n |f| \xrightarrow{w} |f|$ so that $(\tau_n f)$ is wsp (as it is bounded by a weakly convergent sequence) and so for some subsequence $n(j)$, $\tau_{n(j)} f \xrightarrow{w} g$ (say). But then

$$|f| \pm f \xleftarrow{w} T_{n(j)}(|f| \pm f) \leqslant \tau_{n(j)}(|f| \pm f) \xrightarrow{w} |f| \pm g$$

which shows that $g = f$. So $\tau_n f \xrightarrow{w} f$.

Proof of Theorem 1.2. Without loss of generality we may again assume that E has a weak unit e with $\|e\| = 1$ and that $T_n e \xrightarrow{s} e$.

Fix $f \in N$. By Proposition 3.2 N is a sublattice of E which therefore contains $|f|$ so that $T_n |f| \xrightarrow{w} |f|$.

We first show that $\tau_n e \xrightarrow{s} e$.

Since $T_n e \xrightarrow{s} e$ then $|T_n e| \xrightarrow{s} e$ and

$$\|\tau_n e - e\| \leqslant \|\tau_n e - |T_n e|\| + \||T_n e| - e\|$$

which means that we need only show that

$$\lim_n \|\tau_n e - |T_n e|\| = 0. \tag{3.2}$$

But for $x \geqslant 0$,

$$\tau x = \sup_{|y| \leqslant x} |Ty| \geqslant |Tx| \geqslant 0$$

so that by the AL property

$$\begin{aligned} \| \tau_n e - |T_n e| \, \| &= \| \tau_n e \| - \| \, |T_n e| \, \| \\ &\leqslant 1 - \| \, |T_n e| \, \| \qquad \text{(as } \tau_n \text{ is a contraction)} \\ &\to 0 \end{aligned}$$

since $|T_n e| \xrightarrow{s} e$ implies that $\| \, |T_n e| \, \| \to \|e\| = 1$. This gives equation (3.2).

N itself can now be viewed as an AL space with a weak unit. It is therefore representable as the L_1 space of a compact measure space X ([3], p. 114) where e becomes the constant function 1.

To show that $\tau_n f \xrightarrow{s} f$ for all $f \in N$, it suffices to consider characteristic functions χ_E for E a measurable subset of X (because finite linear combinations of such are norm dense in N). Adapting Meir's argument in ([2], Corollary) we have

$$\tau_n \chi_E - \chi_E = (\tau_n 1 - 1) \cdot \chi_E - (\tau_n \chi_{\bar{E}}) \cdot \chi_E + (\tau_n \chi_E) \cdot \chi_{\bar{E}}$$

(where $\bar{E}$ is the complement of E) so that

$$\| \tau_n \chi_E - \chi_E \| \leqslant \| \tau_n 1 - 1 \| + \int (\tau_n \chi_{\bar{E}}) \cdot \chi_E + \int (\tau_n \chi_E) \cdot \chi_{\bar{E}}$$

The first term on the right converges to zero by the previous result and the other two converge to zero by Proposition 3.2 (ii).

Finally we show that

$$T_n f \xrightarrow{s} f \qquad \text{for all} \quad f \in N.$$

Let $f \in N$, $f \geqslant 0$. Then

$$\begin{aligned} 0 \leqslant (\tau_n - T_n) f = (\tau_n - T_n)(f - f \wedge me) \\ + (\tau_n - T_n)(f \wedge me). \end{aligned}$$

Choosing m large so that $\| f - f \wedge me \|$ is small and noting that for fixed m

$$(\tau_n - T_n)(f \wedge me) \leqslant (\tau_n - T_n) \, me \xrightarrow{s} 0 \qquad \text{we have} \quad \| \tau_n f - T_n f \| \to 0.$$

Hence

$$T_n f \xrightarrow{s} f.$$

Applying this result to $|f| \pm f$ we have $T_n f \xrightarrow{s} f$ for all $f \in N$.

This proves the theorem.

4. CONSEQUENCES OF THE KOROVKIN THEOREM

We mention here (without proofs) two straightforward corollaries to Theorem 1.2

COROLLARY 4.1. *Let T be a contraction operator on an L_1 space which has a positive fixed point. Then the set offered points of T is a (closed) sublattice of L_1.*

As an example, T might arise from an infinite matrix acting on $l_1(\mathbf{Z}^+)$ and whose row sums are all 1.

Birkhoff ([1], p. 391) obtained a similar result for transition operators which map probability distributions to probability distributions.

COROLLARY 4.2. *Let N be the subspace of $L_1[0,1]$ spanned by $\{1, x\}$. Then there is no norm 1 projection of $L_1[0,1]$ onto N.*

This generalises a result of Wulbert ([4], Corollary 13) (where he takes N to be spanned by $\{1, x, x^2\}$.

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